Error Estimation for Arnoldi-based Model Order Reduction of MEMS

T. Bechtold, E. B. Rudnyi and J. G. Korvink

IMTEK, University of Freiburg, 79110 Freiburg, Germany
Tel:+49-761-2037389, Fax:+49-761-2037382,bechtold@imtek.de

ABSTRACT

In this paper we present two different, heuristic error estimates for the Pade-type approximation of transfer functions via an Arnoldi algorithm. We first suggest a convergence criterion between two successive reduced models of the order r and $r+1$. We further propose to use the solution of the Lyapunov equations for reduced-order systems as a stop-criterion during iterative model order reduction.
Keywords: error estimate, model order reduction, Arnoldi algorithm, convergence, Hankel singular values.

1 INTRODUCTION

Let us consider a problem of iteratively generating a reduced-order model for a stable linear time-invariant state-space system (1) and evaluate it. For simplicity we will assume a Single-Input-Single-Output setup in the following:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + \boldsymbol{b}u(t) \\ y(t) &= \boldsymbol{c}^T \cdot x(t) \end{aligned} \tag{1}$$

Here $A \in R^{nxn}$ is large, possibly sparse system matrix, $\boldsymbol{b} \in R^n$ and $\boldsymbol{c} \in R^n$ are the control and the observation vectors, respectively. Large-scale systems arise, for instance, from the finite-element modeling of MEMS devices, and have to be compacted for subsequent system-level simulation. For a completely automatic model order reduction (MOR), it is essential to have an explicit estimate of the approximation error, which will determine the order of the reduced model required to achieve a desired accuracy.

Well-established model reduction methods, such as balanced truncations (BT) [1] begin by solving the Lyapunov equations:

$$AP + PA^T = -\boldsymbol{b}\boldsymbol{b}^T, \quad A^TQ + QA = -\boldsymbol{c}\boldsymbol{c}^T \tag{2}$$

and then computing the Hankel singular values (HSV) defined as:

$$\sigma_i = \sqrt{\lambda_i(P \cdot Q)}, i = 1, \ldots, n \tag{3}$$

Those yield a global error bound:

$$\left\| G - G_r \right\|_\infty \le 2(\sigma_{r+1} + \ldots + \sigma_n) \tag{4}$$

between the transfer function $G(s)$ of the original state-space model (1) and the transfer function $G_r(s)$ of it's reduced order-r model (usually $r \ll n$). Hereby, the reduced model is obtained by using projectors originating from the solutions to (2). However, it is not realistic to solve Lyapunov equations by dense matrix techniques for the problems containing more than say, 2000 degrees of freedom. Hence, for higher dimensional problems, Krylov-subspace methods [2] or sequential strategies [3] must be used. However, the question of an effective stop criterion for these methods is still open. To our knowledge, only local (single-frequency) error estimates for these methods have been suggested so far [4], [5]. We propose two different "heuristic" error estimates for the Pade-type approximation of transfer functions via an Arnoldi algorithm [6]. The idea is either to compute the relative error between the successive reduced order models (a similar suggestion can be found in [7]) or, alternatively, to compute the HSV of the reduced model in each iteration of the Arnoldi algorithm.

We have shown numerically that both estimates work well for two electro-thermal MEMS devices, and present here the results for one of them, a pyrotechnical microthruster [8]. The microthruster device (Fig. 1) is based on the integration of solid fuel with a silicon micromachined structure and is electro-thermally ignited. We have used a two-dimensional model, which after the FE-based spatial discretization results in a linear system (1) of 1071 ordinary differential equations.

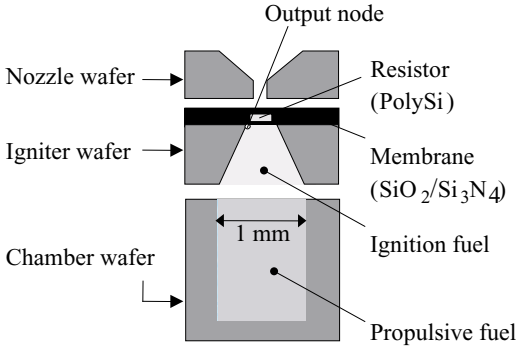

Fig. 1 Microthruster structure.

2 CONVERGENCE OF RELATIVE ERROR

A simple approach to estimate the model error in either the time- or frequency-domain is to compute the difference between two "neighbored" reduced models with order r and $r+1$. Let us define a relative frequency-response error as:

$$\varepsilon_r(s) = \frac{|G(s) - G_r(s)|}{|G(s)|} \tag{5}$$

where $G(s) = \boldsymbol{c}^T (sI - A)^{-1} \boldsymbol{b}$. Let us further define a relative frequency-response error between two successive reduced order models as:

$$\hat{\varepsilon}_r(s) = \frac{|G_r(s) - G_{r+1}(s)|}{|G_r(s)|} \tag{6}$$

We have found that for our test cases $\varepsilon_r(s) \approx \hat{\varepsilon}_r(s)$ for a wide range of frequencies around the expansion point ($s_0 = 0$). The frequency responses of the original model and three reduced order models are shown in Fig. 2.

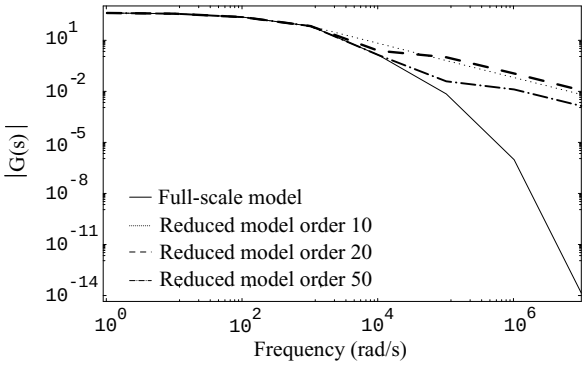

Fig. 2 Frequency response.

Fig. 3 through Fig. 6 compare the true error $\varepsilon_r(s)$ (dashed line) to the estimate (6) (dotted line) for different frequencies.

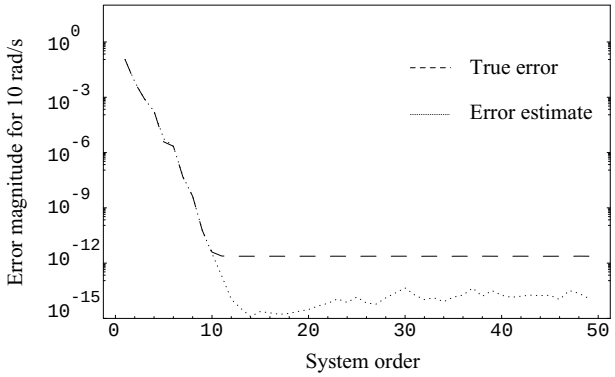

Fig. 3 Error estimate for $\omega = 10 rad/s$.

We can observe two effects: At frequencies up to

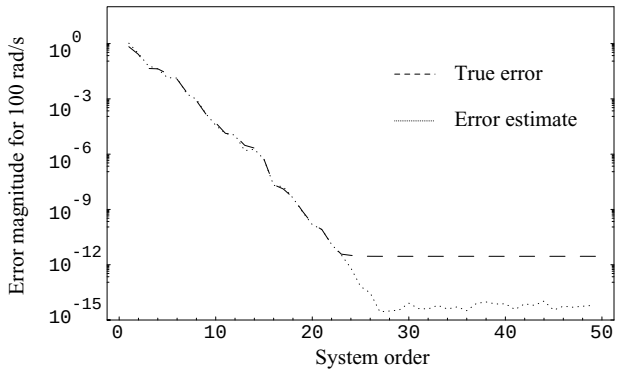

Fig. 4 Error estimate for $\omega = 10^2 rad/s$.

$\omega = 10^4 rad/s$ convergence occur when a certain order of the reduced system is reached (Fig. 3 trough Fig. 5).

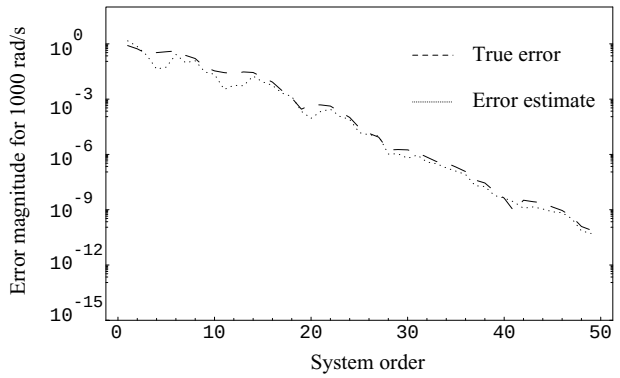

Fig. 5 Error estimate for $\omega = 10^3 rad/s$.

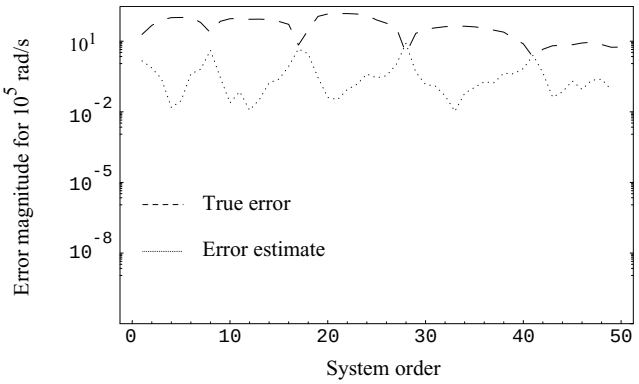

Fig. 6 Error estimate for $\omega = 10^5 rad/s$.

This means that for $\omega = 10 rad/s$ it is not possible to approximate the system better with more than 10 Arnoldi vectors. The minimal error of the approximation is then given through $\hat{\varepsilon}_{10}$. The system order necessary to reach convergence increases towards higher frequencies. Convergence occurs because one presumably reaches the machine's numerical precision. At high frequency, the convergence disappears. Instead, we observe fluctuations (Fig. 6), due to

expanding the transfer function around zero frequency. For an expansion around a higher frequency, we expect to achieve the convergence in Fig. 6 as well.

As already mentioned, this estimate functions also in the time-domain. Let us define a quadratic relative step-response error as:

$$\varepsilon(r) = \sqrt{\frac{\sum_{T=0}^{N\cdot\Delta t}\left(\frac{y(T)-y_r(T)}{y(T)}\right)^2}{N}} \tag{7}$$

where $y(T)$ and $y_r(T)$ are the system outputs of the full and order r reduced system in N discrete time-points spaced Δt apart**.** Let us further define a quadratic relative step-response error between two successive reduced order models as:

$$\hat{\varepsilon}(r) = \sqrt{\frac{\sum_{T=0}^{N\cdot\Delta t}\left(\frac{y_r(T)-y_{r+1}(T)}{y_r(T)}\right)^2}{N}} \tag{8}$$

Again we have that $\varepsilon(r) \approx \hat{\varepsilon}(r)$. Fig. 7 compares the true error $\varepsilon(r)$ (dashed line) to the estimate $\hat{\varepsilon}(r)$ (dotted line) for discrete times between $0s$ and $0.3s$ with $\Delta t = 0.01s$.

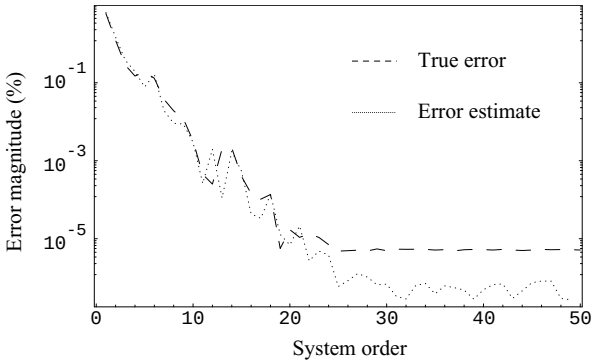

Fig. 7 Time domain estimate.

Note that the errors (7) and (8) are not functions of time, but rather of the system order, and are integrated over a time-range. Hence, they are slightly more expensive to compute than the frequency-response errors.

3 CONVERGENCE OF HANKEL SINGULAR VALUES

Another possibility to approximate the frequency-response error is to modify equation (4) in terms of computing the HSV of the reduced system in each iteration, instead σ_i of the original model. In this way, after i iterations we get a matrix-like structure:

$$H_i = \begin{bmatrix} \hat{\sigma}_{11} & 0 & \dots & 0 \\ \hat{\sigma}_{21} & \hat{\sigma}_{22} & \dots & 0 \\ & \dots & & \\ \hat{\sigma}_{i1} & \hat{\sigma}_{12} & \dots & \hat{\sigma}_{ii} \end{bmatrix} \tag{9}$$

where $\hat{\sigma}_{ij}$ is the j-th HSV of the i-th order reduced model. We have observed that after a number of Arnoldi iterations, the largest $\hat{\sigma}_{ij}$ of the created reduced order models converge towards the HSV of the original model. Fig. 8 shows that for

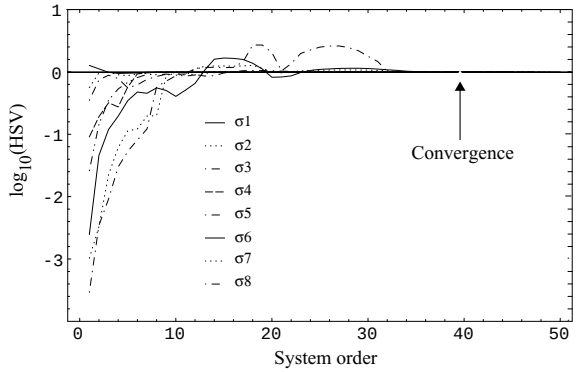

Fig. 8 Largest 8 HSV of the Arnoldi reduced microthruster models (order 1 to 50) normalized to the corresponding HSV of the original model.

the microthruster model, the reduced system of order 40 already reproduces the original 8 largest HSV. Furthermore, in each Arnoldi iteration one new value is added towards the end of the set (provided the $\hat{\sigma}_{ij}$ are sorted in descending order), while the beginning values slowly converge. This means that after a number of iterations we can consider the largest original HSV (those which don't change any more when increasing the reduced system order) as known. Now, assuming the rapid decay of σ_i we can use equation (4) to approximately set a target reduced order for the specified error bound between the transfer functions. To give a simple example, let us set a prescribed error to $\varepsilon = 10^{-1}$ and ask: how many iterations do we have to perform to fulfill this error bound? In other words, when is the sum of the tail of original HSV surely smaller than ε? By observing the matrix H_{40} we see that the eighth HSV has order of magnitude 10^{-2}, and all the previous values have already converged. This means that no new values with order of magnitude bigger than 10^{-2} will appear in further iterations. Hence, at worst case equation (4) leads to too high error:

$$\left|G - G_7\right| \le 2 \cdot (1071 - 7) \cdot 10^{-2} \approx 20 \tag{10}$$

In order to correct this error, we use an HSV decay estimate based on [9] (dotted line in Fig. 9). The estimate [9] is orig-

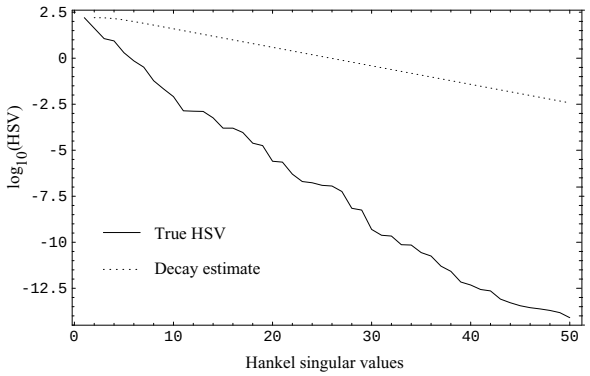

Fig. 9 Decay estimates of HSV for the microthruster.

inally valid for the eigenvalues $\lambda_i(P)$ of one grammian of the symmetric system. It is based only on the knowledge of the condition number κ of A:

$$\frac{\lambda_{k+1}(P)}{\lambda_k(P)} \le \left(\prod_{j=0}^{k-1} \frac{\kappa(A)^{(2j+1)/(2k)} - 1}{\kappa(A)^{(2j+1)/(2k)} + 1} \right)^2 \qquad (11)$$

Since we have observed essentially the same quality of results for the decay estimate of HSV and $\lambda_i(P)$ for the microthruster, we use a formula (11) to correct the right side of inequality (10). By summing all the HSV estimates with an order of magnitude of 10^{-2} or smaller, we get 0.14, which is already near ε. In this way we have got an "indication" that we could already fulfill a prescribed error bound with a reduced system of order 7. Indeed, for the original system it holds:

$$\left| G - G_7 \right| \le 2(\sigma_8 + \ldots + \sigma_{1071}) \; = \; 0.094 \qquad (12)$$

Note once more that G_7 in equation (12) should be computed by BT, and that the difference to the reduced order model computed by Arnoldi must be accounted for as well. For the microthruster model this difference was shown in [10].

4 DISCUSSION

In this paper we have presented two heuristic error estimation possibilities for the Pade-type approximation of a transfer function via the Arnoldi algorithm. Both were tested on electro-thermal MEMS models.

A convergence of relative errors allows an approximation of true model error through the error between successive reduced systems. It seems to function in both the frequency and time domain, and the implementation requires only the additional solution of a small linear equation system in each iteration. It is accurate at frequencies near the expansion point (tested for $s_0 = 0$), but suffers from fluctuations at higher frequencies ($\omega \ge 10^5$ rad/s), and shows a similar local character as the estimate suggested in [4].

An observation of Hankel singular values shows that the HSV of the *i*-th reduced order model are approximations of the HSV of the original model. Hence, the global error estimate (4) valid for control theory methods can be approximated without necessarily computing the original HSV. For this, a decay estimate is necessary. We have shown that the estimate [9], although valid only for symmetric A, could be sufficient.

It is important to say that there is still no theoretical justification of the proposed error estimation techniques at this time, and wether they will work in general is still an open question. Nevertheless, the methods can be considered as an engineering solution, and can be applied to electro-thermal MEMS models. They allow completely automatic order reduction of MEMS, where the time-consuming simulation of the large-scale original model is circumvented.

5 REFERENCES

[1] D. Hinrichsen, H. W. Philippsen, "Modellreduktion mit Hilfe balancierter Realisierungen", *Automatisierungstechnik*, **38**, pp. 416-422, (1990).

[2] Z. Bai, " Krylov subspace techniques for reduced-order modeling of large-scale dynamical systems", *Elsevier Appl. Num. Math.*, **43**, pp. 9-44 (2002).

[3] M. Kammon, F. Wang, J. White, "Generating Nearly Optimally Compact Models from Krylov-Subspace Based Reduced-Order Models", *IEEE Trans. Circuits Syst.*, **47**, pp. 239-248 (2000).

[4] Z. Bai, Q. Ye, "Error Estimation of the Pade Approximation of Transfer Function via the Lanczos Process", *El. Tran. Num. Anal.*, **7**, pp. 1-17 (1998).

[5] I. M. Jaimoukha, E. M. Kasenally, "Oblique Projection Methods for Large Scale Model Reduction", *SIAM J. Matrix Anal. Appl.*, **16**, pp. 602-627 (1995).

[6] L. M. Silveira, M. Kamon, I. Elfadel, J. White, "A Coordinate-transformed Arnoldi Algorithm for Generating Guaranteed Stable Reduced-Order Models of RLC Circuits", Comput. Meth. Appl. Mech. Eng., 169, pp. 377-389, (1999).

[7] E. Grimm, "Krylov projection methods for model reduction ", *Ph.D. Thesis*, Univ. of Illinois at Urbana-Champaign, 1997.

[8] C. Rossi, "Micropropulsion for Space", *Sensors Updates*, **10**, pp. 257-292 (2002).

[9] T. Penzil, "Eigenvalue decay bounds for solutions of Lyapunov equations: the symmetric case", *Systems Control Lett.*, **40**, pp. 139-144, (2000).

[10] T. Bechtold, E. B. Rudnyi, J. G. Korvink, "Automatic Generation of Compact Electro-Thermal Models for Semiconductor Devices", *IEICE Trans. Electron.*, **3**, pp. 459 - 465, (2003).